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Joint quasiprobability distribution for N spin-$\frac{1}{2}$ systems

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Abstract. A quantum system is described by a density matrix which determines the joint probabilities for simultaneous eigenstates of its observables. Since there do not exist any simultaneous eigenstates of non-commuting observables, the density matrix cannot provide any information about the joint probability of their eigenstates. The quasiprobability distributions (QPDs) constructed by using the density matrix provide a means of determining these joint probabilities. Here the QPDs for a system of N spin-$\frac{1}{2}$ are constructed and the joint probabilities for their components determined.

1. Introduction

The density matrix provides a complete description of a quantum system by determining the occupation probability of the eigenstates and the probabilities of transitions between these states. It also determines the joint probabilities for simultaneous eigenstates of different observables. Since there do not exist any simultaneous eigenstates of non-commuting observables, the density matrix cannot provide any information about joint probability of occupation of the eigenstates of non-commuting observables. The importance of the joint probabilities in the context of a spin-$\frac{1}{2}$ has been highlighted by Feynman [1] in terms of a *Gedanken* two-slit interference experiment. Scully *et al* [2] have recently proposed a micromaser scheme for realizing that experiment. The relationship between the joint probabilities for two spin-$\frac{1}{2}$ and Bell's inequality has been reported by Wigner [3].

How can one then determine the joint probability for non-commuting observables? *The answer lies, of course, in building a description of the quantum system in terms of* commuting, i.e. c-number, variables. Since the quantum predictions are probabilistic, the c-number equivalence will be in terms of probability distribution functions. That approach, initiated by Wigner [4], has been the key, not only as an operational tool, but also to the understanding of the relationship between classical and quantum mechanics. Since the quantum indeterminism is not the same as the classical statistical one, the c-number distribution functions may not always possess the properties of a classical probability distribution function. Those are, therefore, referred to as the quasiprobability distributions (QPDs). The QPDs may, for example, be negative. However, they should, and do, give meaningful results for observable quantities.

The QPDs have been constructed for systems of spins [5–7]. There, however, they are in terms of the variables which have no direct relationship with the eigenvalues of the observables of interest, namely the projection of the spin in an arbitrary direction. Those QPDs, therefore, are not very useful in answering the questions related to the joint probability for the components along different directions being in particular eigenstates. Those questions

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are, however, answered by the QPD of Scully and co-workers [8,9]. Their approach has been used for determining the joint probabilities for two components of spin-$\frac{1}{2}$ [2]. The expressions for the probabilities so obtained differ from those of Feynman [1]. However, as pointed out in [2], Feynman introduced his expressions without providing any derivation or motivation.

In this paper we introduce a QPD suitable for determining the joint probabilities for the components of the spins in a system of N spin-$\frac{1}{2}$. Our approach is to describe a spin-$\frac{1}{2}$ in terms of three two-state random variables corresponding to three components of the three-dimensional spin vector $\boldsymbol{\sigma}$. The components need not be orthogonal. The joint probabilities are then determined in terms of the averages of the random variables. The distribution so constructed is the desired QPD if the c-number averages are appropriately identified with the quantum expectation values. The identification of the averages of single random variables with the operator averages is, of course, straightforward but associating the averages involving products of random variables with the operator products is, as is well known, ambiguous. The choice of different rules of association leads to different QPDs. However, in our approach, that choice is made at the end, which is unlike the usual approach [8,9] where the selection of the ordering is made at the beginning. This is because we proceed by constructing a classical distribution to mimic the quantum spins, whereas the usual approach is to construct the QPD by defining a generating function in terms of the averages of operators with respect to the given quantum state, therefore necessitating the choice of the ordering in the definition of the generating function. In the approach presented here one works only with c-numbers to arrive at the expression for the probability distribution in terms of c-number averages, which are identified with the operator averages to obtain the QPD. Moreover, our procedure admits a simple generalization to a system of N spin-$\frac{1}{2}$. We, of course, recover the results of Scully *et al* [2] as a special case. Interestingly, we find that Feynman's definition for the joint probabilities for *two* orthogonal components has, in fact, the structure of our expressions for the joint probability of *three* orthogonal components.

Consider first a spin-$\frac{1}{2}$ system described quantum mechanically by the spin operator $\hat{\boldsymbol{\sigma}}$ whose components $\hat{\sigma}_{e_a} = \boldsymbol{e}_a \cdot \hat{\boldsymbol{\sigma}}$, $\hat{\sigma}_{e_b} = \boldsymbol{e}_b \cdot \hat{\boldsymbol{\sigma}}$ along the directions $\boldsymbol{e}_a$, $\boldsymbol{e}_b$ ($|\boldsymbol{e}_a| = |\boldsymbol{e}_b| = 1$) obey the anticommutation relation

$$\hat{\sigma}_{e_a}\hat{\sigma}_{e_b} + \hat{\sigma}_{e_b}\hat{\sigma}_{e_a} = \frac{\boldsymbol{e}_a \cdot \boldsymbol{e}_b}{2} \tag{1}$$

and the commutation relation

$$[\hat{\sigma}_{e_a}, \hat{\sigma}_{e_b}] = \mathrm{i}(\boldsymbol{e}_a \times \boldsymbol{e}_b) \cdot \hat{\boldsymbol{\sigma}}. \tag{2}$$

Quantum mechanically, the component of the spin-$\frac{1}{2}$ along any direction can have the values $\pm\frac{1}{2}$ whose probability of occurrence is determined by the state of the system.

We construct a classical analogue of the spin-$\frac{1}{2}$ by assuming that its projection along any direction is a random variable capable of assuming two values $\pm\frac{1}{2}$. The system is described completely in terms of three random variables a, b and c which are the projections of the three-dimensional spin vector along the directions $\boldsymbol{e}_a$, $\boldsymbol{e}_b$ and $\boldsymbol{e}_c$, respectively. The classical distribution function $f(a, b, c)$ of the spin is then evidently given by

$$f(a,b,c) = \sum_{i,j,k=1}^{2} p(\epsilon_i, \epsilon_j, \epsilon_k)\delta\left(a - \frac{\epsilon_i}{2}\right)\delta\left(b - \frac{\epsilon_j}{2}\right)\delta\left(c - \frac{\epsilon_k}{2}\right) \qquad \epsilon_1 = +1 \quad \epsilon_2 = -1 \tag{3}$$

where $p(\epsilon_i, \epsilon_j, \epsilon_k)$, $i, j, k = 1, 2$ ($\epsilon_1 = +1, \epsilon_2 = -1$), is the joint probability for the variables a, b and c to have values $\epsilon_i/2$, $\epsilon_j/2$ and $\epsilon_k/2$, respectively. Note that there are eight unknown joint probabilities $p(\epsilon_i, \epsilon_j, \epsilon_k)$ and eight independent averages corresponding to the averages of the three random variables a, b and c; the averages of all their possible products and the normalization of the probabilities. The $p(\epsilon_i, \epsilon_j, \epsilon_k)$ can therefore be determined in terms of those averages by solving eight coupled equations obtained from (3). Those equations break up in two groups of four equations each. The solution of those equations leads to the following expressions for the joint probabilities

$$p(\epsilon_i, \epsilon_j, \epsilon_k) = \frac{1}{2^3}[1 + 2\epsilon_i\langle a\rangle + 2\epsilon_j\langle b\rangle + 2\epsilon_k\langle c\rangle + 4\epsilon_i\epsilon_j\langle ab\rangle$$
$$+4\epsilon_i\epsilon_k\langle ac\rangle + 4\epsilon_j\epsilon_k\langle bc\rangle + 8\epsilon_i\epsilon_j\epsilon_k\langle abc\rangle] \quad (4)$$

where $\langle A\rangle$ denotes the average of A with respect to $f(a, b, c)$. The distribution $f(a, b, c)$ would represent a given quantum state of the spin-$\frac{1}{2}$ if the averages in (4) are identified with those of the spin operators and their products in that state. It is straightforward to identify $\langle a\rangle$, $\langle b\rangle$ and $\langle c\rangle$ with $\langle\hat{\sigma}_{e_a}\rangle$, $\langle\hat{\sigma}_{e_b}\rangle$ and $\langle\hat{\sigma}_{e_c}\rangle$, respectively. However, since the spin operators do not commute, the identification of the average of the products of a, b and c with those of the spin operators is ambiguous. The choice of different rules of association of the products of classical variables with those of the quantum operators leads to different quasiprobability distribution functions. One can, for example, choose the 'Wigner-like' symmetric rule of correspondence whereby, invoking (1),

$$\langle ab\rangle \to \tfrac{1}{2}(\langle\hat{\sigma}_{e_a}\hat{\sigma}_{e_b} + \hat{\sigma}_{e_b}\hat{\sigma}_{e_a}\rangle) = \frac{e_a \cdot e_b}{4} \qquad \text{etc}$$
$$\langle abc\rangle \to \tfrac{1}{12}[\langle\hat{\sigma}_{e_a}(\hat{\sigma}_{e_b}\hat{\sigma}_{e_c} + \hat{\sigma}_{e_c}\hat{\sigma}_{e_b})\rangle + \langle(\hat{\sigma}_{e_b}\hat{\sigma}_{e_c} + \hat{\sigma}_{e_c}\hat{\sigma}_{e_b})\hat{\sigma}_{e_a}\rangle$$
$$+(a \to b, b \to c, c \to a) + (a \to c, c \to b, b \to a)]$$
$$= \tfrac{1}{6}[(e_b \cdot e_c)\langle\hat{\sigma}_{e_a}\rangle + (e_a \cdot e_c)\langle\hat{\sigma}_{e_b}\rangle + (e_a \cdot e_b)\langle\hat{\sigma}_{e_c}\rangle]. \quad (5)$$

By substituting (5) in (4) we obtain the joint QPD *for the three spin components of a spin-$\frac{1}{2}$* as

$$p(\epsilon_i, \epsilon_j, \epsilon_k) = \frac{1}{2^3}\bigg[1 + 2\epsilon_i\langle\hat{\sigma}_{e_a}\rangle + 2\epsilon_j\langle\hat{\sigma}_{e_b}\rangle + 2\epsilon_k\langle\hat{\sigma}_{e_c}\rangle$$
$$+\epsilon_i\epsilon_j e_a \cdot e_b + \epsilon_j\epsilon_k e_b \cdot e_c + \epsilon_i\epsilon_k e_c \cdot e_a$$
$$+\frac{4\epsilon_i\epsilon_j\epsilon_k}{3}\{(e_b \cdot e_c)\langle\hat{\sigma}_{e_a}\rangle + (e_a \cdot e_c)\langle\hat{\sigma}_{e_b}\rangle + (e_a \cdot e_b)\langle\hat{\sigma}_{e_c}\rangle\}\bigg]. \quad (6)$$

As a special case, note that the joint probabilities for two components, say, along e_a and e_b are given by

$$p(\epsilon_i, \epsilon_j) = \tfrac{1}{4}[1 + 2\epsilon_i\langle\hat{\sigma}_{e_a}\rangle + 2\epsilon_j\langle\hat{\sigma}_{e_b}\rangle + e_a \cdot e_b] \quad (7)$$

which are the same as the ones derived by Scully *et al* [2] in the symmetric ordering of the operators. Note that the joint probabilities for two components proposed by Feynman [1, 2] involves the expectation values of three components. However, note from equations (6) and (7) that it is the joint probability of three components that involves the expectation value of three components, whereas the joint probability of two components involves the expectation values of only these two components.

We have thus determined the complete joint QPD for a spin-$\frac{1}{2}$. One can easily identify the values of the parameters defining a state of the spin-$\frac{1}{2}$ for which one or more joint probabilities become negative. The importance of negative probabilities in the context of Young's double slit experiment has been discussed in [1, 2].

Next, we determine the joint QPD for two spin-$\frac{1}{2}$ systems described in terms of the classical variables (a_1, b_1, c_1) and (a_2, b_2, c_2) which are projections of the two spins along the directions $(e_{a_1}, e_{b_1} e_{c_1})$ and $(e_{a_2}, e_{b_2} e_{c_2})$, respectively. The classical distribution function of the combined system for the variables $(a_1, c_1; a_2, c_2)$ irrespective of the state of b_1 and b_2, is clearly given by

$$f(a_1, c_1; a_2, c_2) = \sum_{i,j,k,l=1}^{2} p(\epsilon_i, \epsilon_j; \epsilon_k, \epsilon_l)\left[\delta\left(a_1 - \frac{\epsilon_i}{2}\right)\delta\left(c_1 - \frac{\epsilon_j}{2}\right)\delta\left(a_2 - \frac{\epsilon_k}{2}\right)\delta\left(c_2 - \frac{\epsilon_l}{2}\right)\right]. \tag{8}$$

Sixteen joint probabilities $p(\epsilon_i, \epsilon_j; \epsilon_k, \epsilon_l)$ can be determined in terms of the averages by solving sixteen coupled equations for the averages of a_1, c_1, a_2, c_2; the averages of all of their possible products and the normalization of the probabilities. These equations reduce to four decoupled sets each of four equations. The solution of these equations leads to

$$\begin{aligned} p(\epsilon_i, \epsilon_j; \epsilon_k, \epsilon_l) = \frac{1}{2^4}[1 &+ 2\epsilon_i\langle a_1\rangle + 2\epsilon_j\langle c_1\rangle + 2\epsilon_k\langle a_2\rangle + 2\epsilon_l\langle c_2\rangle \\ &+ 4\epsilon_i\{\epsilon_j\langle a_1c_1\rangle + \epsilon_k\langle a_1a_2\rangle + \epsilon_l\langle a_1c_2\rangle\} + 4\epsilon_j\{\epsilon_k\langle c_1a_2\rangle + \epsilon_l\langle c_1c_2\rangle\} + 4\epsilon_k\langle a_2c_2\rangle \\ &+ 8\epsilon_i\{\epsilon_j\epsilon_k\langle a_1c_1a_2\rangle + \epsilon_j\epsilon_l\langle a_1c_1c_2\rangle + \epsilon_k\epsilon_l\langle a_1a_2c_2\rangle\} \\ &+ \epsilon_j\epsilon_k\epsilon_l\langle c_1a_2c_2\rangle + 16\epsilon_i\epsilon_j\epsilon_k\epsilon_l\langle a_1c_1a_2c_2\rangle]. \end{aligned} \tag{9}$$

By identifying the c-number averages in (9) with the expectation values as in the case of one spin-$\frac{1}{2}$, we obtain the joint QPD for two components of each of the two spin-$\frac{1}{2}$. A special case of interest is the joint probability $p(\epsilon_i, \epsilon_j)$ for the two spins to have the values $\epsilon_i/2$ and $\epsilon_j/2$ in the directions e_a and e_b respectively. From (9) it straightforwardly follows that

$$p(\epsilon_i, \epsilon_j) = \tfrac{1}{4}[1 + 2\epsilon_i\langle\hat{\sigma}^{(1)}_{e_a}\rangle + 2\epsilon_j\langle\hat{\sigma}^{(2)}_{e_b}\rangle + 4\epsilon_i\epsilon_j\langle\hat{\sigma}^{(1)}_{e_a}\hat{\sigma}^{(2)}_{e_b}\rangle]. \tag{10}$$

If the two spins are in the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}, \tfrac{1}{2}\rangle] \tag{11}$$

where the states $|\pm\frac{1}{2}, \pm\frac{1}{2}\rangle$ are the eigenstates of $|\sigma^z_{1,2}\rangle$ then $\langle\sigma^{(1)}_{e_a}\rangle = \langle\sigma^{(1)}_{e_b}\rangle = 0$ and

$$\langle\sigma^{(1)}_{e_a}\sigma^{(2)}_{e_b}\rangle = \frac{-e_a \cdot e_b}{4} \tag{12}$$

so that

$$p(\epsilon_i, \epsilon_j) = \tfrac{1}{4}[1 - \epsilon_i\epsilon_j e_a \cdot e_b] \tag{13}$$

which is the same as the one derived by the other methods (see, for example, [6, 7]). Note that the joint probability here is for the observables for two different spins, i.e. for commuting observables. Hence, the usual density matrix approach is also adequate for determining that joint probability. We will see that these joint probabilities are of particular interest in determining the correctness or otherwise of the classical statistical description of the quantum theory.

We can derive also the most general QPD, i.e. for the joint probability $p(\epsilon_i, \epsilon_j, \epsilon_k; \epsilon_l, \epsilon_m, \epsilon_n)$ of the three components each of the two spins. It is found to be given by

$$p(\epsilon_i, \epsilon_j, \epsilon_k; \epsilon_l, \epsilon_m, \epsilon_n) = \frac{1}{2^6}\left[1 + \sum_{m=1}^{6} 2^m\langle\{\text{product of } m \text{ of } \epsilon_i a_1, \epsilon_j b_1, \epsilon_k c_1, \epsilon_l a_2, \epsilon_m b_2, \epsilon_n c_2\}\rangle\right]. \tag{14}$$

The probabilities (14) can be used to arrive at Bell's inequality, which involves the measurement of the spin components along three directions, and show that its violation implies that the probabilities can be negative. For, if p_{ab} is the joint probability for finding the spin-1 to be along the direction e_a and the spin-2 to be along e_b then by using the equation

$$p_{ab} = \sum_{j,k,l,n} p(+, \epsilon_j, \epsilon_k; \epsilon_l, +, \epsilon_n) \tag{15}$$

where $p(\epsilon_i, \epsilon_j, \epsilon_k; \epsilon_l, \epsilon_m, \epsilon_n)$ is the probability that the spin-1 has the values $\epsilon_i/2, \epsilon_j/2, \epsilon_k/2$ and the spin-2 has the values $\epsilon_l/2, \epsilon_m/2, \epsilon_n/2$ along the directions e_a, e_b, e_c, respectively, it then follows that

$$p_{ab} + p_{bc} - p_{ac} = \sum [p(+, \epsilon_j, \epsilon_k; \epsilon_l, +, -) + p(-, +, \epsilon_l; \epsilon_m, \epsilon_m, +) \\ + p(+, +, \epsilon_l; \epsilon_m, +, +) - p(+, -, \epsilon_l; \epsilon_m, -, +)]. \tag{16}$$

Note from (13) that the probability for the two spins to be aligned parallel to the same direction is zero, i.e.

$$\sum p(\epsilon, \epsilon_j, \epsilon_k; \epsilon, \epsilon_m, \epsilon_n) = \sum p(\epsilon_i, \epsilon, \epsilon_k; \epsilon_l, \epsilon, \epsilon_n) = \sum p(\epsilon_i, \epsilon_j, \epsilon; \epsilon_l, \epsilon_m, \epsilon) = 0. \tag{17}$$

Hence it follows that if the joint probabilities are classical, i.e. positive, then the individual terms in (17) should vanish. In that case (16) reduces to

$$p_{ab} + p_{bc} - p_{ac} = p(+, -, +; -, +, -) + p(-, +, -; +, -, +) \tag{18}$$

which, for classical positive distributions, would imply [3]

$$p_{ab} + p_{bc} - p_{ac} > 0. \tag{19}$$

The inequality (19) is Bell's inequality whose violation would, therefore, imply that the underlying joint distributions are negative, i.e. non-classical. By using the expression (13) in (19) it is clear that the inequality is violated, for example, if the three vectors e_a, e_b, e_c are coplanar such that the vector e_b lies between the other two at an angle of $\pi/3$ with each of the others.

The method outlined above can be generalized to N spin-$\frac{1}{2}$ systems. It is not difficult to show that the joint probability $p(\epsilon_{i_1}, \epsilon_{j_1}, \epsilon_{k_1}; \epsilon_{i_2}, \epsilon_{j_2}, \epsilon_{k_2}; \ldots \epsilon_{i_N}, \epsilon_{j_N}, \epsilon_{k_N})$ for the spins $1, 2, \ldots, N$ to have components $(\epsilon_{i_1}/2, \epsilon_{j_1}/2, \epsilon_{k_1}/2); (\epsilon_{i_2}/2, \epsilon_{j_2}/2, \epsilon_{k_2}/2); \ldots (\epsilon_{i_N}/2, \epsilon_{j_N}/2, \epsilon_{k_N}/2)$ along the directions $(e_{a_1}, e_{b_1}, e_{c_1}); (e_{a_2}, e_{b_2}, e_{c_2}); \ldots (e_{a_N}, e_{b_N}, e_{c_N})$, respectively, is given by

$$p(\epsilon_{i_1}, \epsilon_{j_1}, \epsilon_{k_1}; \epsilon_{i_2}, \epsilon_{j_2}; \epsilon_{k_2}; \ldots \epsilon_{i_N}, \epsilon_{j_N}, \epsilon_{k_N}) = \frac{1}{2^{3N}} \Bigg[1 + \sum_{m}^{3N} 2^m \langle \{\text{product of m of} \\ \epsilon_{i_1} a_1, \epsilon_{j_1} b_1, \epsilon_{k_1} c_1, \epsilon_{i_2} a_2, \epsilon_{j_2} b_2, \epsilon_{k_2} c_2, \ldots, \epsilon_{i_N} a_N, \epsilon_{j_N} b_N, \epsilon_{k_N} c_N\} \rangle \Bigg]. \tag{20}$$

The explicit expression for the joint probability, for example, for one component of each of the spins is given by

$$p(\epsilon_{i_1}; \epsilon_{i_2}; \ldots \epsilon_{i_N}) = \frac{1}{2^N} [1 + 2 \sum_{k=1}^{N} \epsilon_{i_k} \langle a_k \rangle + 2^2 \sum_{k(l=1}^{N} \epsilon_{i_k} \epsilon_{i_l} \langle a_k a_l \rangle + \cdots \\ + 2^m \sum_{k_1 < k_2 < \cdots < k_m = 1}^{N} \epsilon_{i_{k_1}} \epsilon_{i_{k_2}} \ldots \epsilon_{i_{k_m}} \langle a_{k_1} a_{k_2} \ldots a_{k_m} \rangle \\ + \cdots + 2^N \epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_N} \langle a_1 a_2 \ldots a_N \rangle]. \tag{21}$$

The averages in (21) refer to different spins. Hence, that result is derivable also by using the density matrix approach.

The expressions derived in the foregoing can be easily used to obtain the joint probabilities for spin-S, because the spin-S corresponds to the fully symmetric state of $N(=2S)$ identical spin-$\frac{1}{2}$ systems. We can, thus, of course, determine the QPD for spin-S from the expressions written here in terms of individual spin operators but it will be interesting to be able to write the joint probability in terms of the collective operators.

In conclusion, we have obtained a QPD for the joint probabilities for the components of the spins in a system of N spin-$\frac{1}{2}$ in terms of the expectation values of spin operators. The joint probability involving $m \leqslant 3$ components of each of the spins contains expectation values of the products of up to mN operators.

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